

# Cohomology of $\mathfrak{osp}(2|2)$ acting on the spaces of linear differential operators on the superspace $\mathbb{R}^{1|2}$

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## Abstract

We compute the first differential cohomology of the orthosymplectic Lie superalgebra  $\mathfrak{osp}(2|2)$  with coefficients in the superspace of linear differential operators acting on the space of weighted densities on the  $(1, 2)$ -dimensional real superspace. We also compute the same, but  $\mathfrak{osp}(1|2)$ -relative, cohomology. We explicitly give 1-cocycles spanning these cohomology. This work is a simplest generalization of a result by Basdouri and Ben Ammar [Cohomology of  $\mathfrak{osp}(1|2)$  with coefficients in  $\mathfrak{D}_{\lambda, \mu}$ . Lett. Math. Phys. **81**, 239–251 (2007)].

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## 1 Introduction

The space of weighted densities with weight  $\lambda$  (or  $\lambda$ -densities) on  $\mathbb{R}$ , denoted by:

$$\mathcal{F}_\lambda = \left\{ f(dx)^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle  $(T^*\mathbb{R})^{\otimes \lambda}$  for positive integer  $\lambda$ . Let  $\text{Vect}(\mathbb{R})$  be the Lie algebra of all vector fields  $X_F = F \frac{d}{dx}$  on  $\mathbb{R}$ , where  $F \in C^\infty(\mathbb{R})$ . The *Lie derivative*  $L_D$  along the vector field  $D$  makes  $\mathcal{F}_\lambda$  a  $\text{Vect}(\mathbb{R})$ -module for any  $\lambda \in \mathbb{R}$ :

$$L_{X_F}(f(dx)^\lambda) = L_{X_F}^\lambda(f)(dx)^\lambda \quad \text{with} \quad L_{X_F}^\lambda(f) = Ff' + \lambda fF', \quad (1.1)$$

where  $f', F'$  are  $\frac{df}{dx}, \frac{dF}{dx}$ . On the space  $\mathcal{D}_{\lambda, \mu}$  of differential operators  $\mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$  a  $\text{Vect}(\mathbb{R})$ -module structure is given by the formula:

$$X_F \cdot A = L_{X_F}^\mu \circ A - A \circ L_{X_F}^\lambda, \quad (1.2)$$

for any differential operator  $A : f(dx)^\lambda \mapsto (Af)(dx)^\mu$ .

Lecomte, in [11], found the cohomology  $H_{\text{diff}}^1(\mathfrak{sl}(2), \mathcal{D}_{\lambda, \mu})$  and  $H_{\text{diff}}^2(\mathfrak{sl}(2), \mathcal{D}_{\lambda, \mu})$ , where  $\mathfrak{sl}(2)$  is realized as the Lie subalgebra of  $\text{Vect}(\mathbb{R})$  spanned by  $\{X_1, X_x, X_{x^2}\}$  and where  $H_{\text{diff}}^*$

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denotes the differential cohomology; that is, only cochains given by differential operators are considered. These spaces appear naturally in the problem of describing the deformations of the  $\mathfrak{sl}(2)$ -module  $\mathcal{S}_{\mu-\lambda} = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\mu-\lambda-k}$ , the space of symbols of differential operators of  $D_{\lambda,\mu}$ . More precisely, the elements of  $H^1(\mathfrak{sl}(2), V)$  classify the infinitesimal deformations of a  $\mathfrak{sl}(2)$ -module  $V$  and the obstructions to integrability of a given infinitesimal deformation of  $V$  are elements of  $H^2(\mathfrak{sl}(2), V)$  (for examples, see [1, 2, 5, 12]).

Now, we can study the corresponding super structures. More precisely, we consider the superspace  $\mathbb{R}^{1|n}$  equipped with a contact 1-form  $\alpha_n$ , and introduce the superspace  $\mathfrak{F}_{\lambda}^n$  of  $\lambda$ -densities on the superspace  $\mathbb{R}^{1|n}$ . The spaces  $\mathfrak{F}_{\lambda}^n$  are modules over  $\mathcal{K}(n)$ , the Lie superalgebra of contact vector fields on  $\mathbb{R}^{1|n}$ ; the space  $\mathfrak{D}_{\lambda,\mu}^n$  of differential operators  $\mathfrak{F}_{\lambda}^n \rightarrow \mathfrak{F}_{\mu}^n$  is, naturally, a  $\mathcal{K}(n)$ -module. The spaces  $H_{\text{diff}}^i(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}^n)$  for  $i = 1$  and  $2$  need to be computed in order to describe deformations of the  $\mathfrak{osp}(1|2)$ -module  $\mathfrak{S}_{\mu-\lambda}^n = \bigoplus_{k \geq 0} \mathfrak{F}_{\mu-\lambda-\frac{k}{2}}^n$ , a super analogue of  $\mathcal{S}_{\mu-\lambda}$ , see [9].

In [3], Basdouri and Ben Ammar studied this question for  $n = 1$ . In this case,  $\mathfrak{sl}(2)$  is replaced by the Lie superalgebra  $\mathfrak{osp}(1|2)$  naturally realized as a subalgebra of  $\mathcal{K}(1)$ .

Since there seems to be no conceptual difference in the setting or results obtained in the study of the cohomology of  $\mathfrak{osp}(n|2)$  acting on the spaces of linear differential operators on the superspace  $\mathbb{R}^{1|n}$  for  $n$  considered so far ( $0, 1$  and  $2$  in this paper), the point is not to treat in further articles the cases  $n = 3$  and so on. The point is that the behavior and certain properties of the Lie superalgebras  $\mathfrak{osp}(n|2)$  and  $\mathcal{K}(n)$  are similar for  $n < 4$  ([8]); the cases  $n = 0$  and  $n = 1$  are particularly close. However, in several questions, the case  $n = 2$  is exceptional due to an occasional isomorphism  $\mathcal{K}(n) \simeq \text{Vect}(\mathbb{R}^{1|1})$  ([10]), and one never knows *a priori* which type of questions will make a given particular  $n$  exceptional. We can expect that the properties of  $\mathfrak{osp}(n|2)$  and  $\mathcal{K}(n)$  become uniform only for  $n > 6$ . So somebody has to perform all the calculations in the hope to find an interesting result (such, for example, as mentioned in Subsection 4.3).

In this paper we consider the case  $n = 2$ . That is, we consider the orthosymplectic Lie superalgebra  $\mathfrak{osp}(2|2)$  naturally realized as a subalgebra of  $\mathcal{K}(2)$ . We compute here  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda,\mu}^2)$  and  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}^2)$ . Moreover, we give explicit formulae for all the nontrivial 1-cocycles. These spaces arise in the classification of infinitesimal deformations of the  $\mathfrak{osp}(2|2)$ -module  $\mathfrak{S}_{\mu-\lambda}^2 = \bigoplus_{k \geq 0} \mathfrak{F}_{\mu-\lambda-\frac{k}{2}}^2$ . We hope to be able to describe in the future all the deformations of this module.

## 2 Definitions and Notation

Recall that  $C^\infty(\mathbb{R}^{1|2})$  consists of elements of the form:

$$F(x, \theta_1, \theta_2) = f_0(x) + f_1(x)\theta_1 + f_2(x)\theta_2 + f_{12}(x)\theta_1\theta_2,$$

where  $f_0, f_1, f_2, f_{12} \in C^\infty(\mathbb{R})$ , and where  $x$  is the even indeterminate,  $\theta_1$  and  $\theta_2$  are odd indeterminates, i.e.,  $\theta_i\theta_j = -\theta_j\theta_i$ . Let  $|F|$  be the parity of a homogeneous function  $F$ . Let

$$\text{Vect}(\mathbb{R}^{1|2}) = \left\{ F_0\partial_x + F_1\partial_1 + F_2\partial_2 \mid F_i \in C^\infty(\mathbb{R}^{1|2}) \right\},$$

where  $\partial_i = \frac{\partial}{\partial \theta_i}$ . Let  $\mathcal{K}(2)$  be the Lie superalgebra of contact vector fields on  $\mathbb{R}^{1|2}$ :

$$\mathcal{K}(2) = \{ X \in \text{Vect}(\mathbb{R}^{1|2}) \mid \text{there exists } F \in C^\infty(\mathbb{R}^{1|2}) \text{ such that } \mathfrak{L}_X(\alpha_2) = F\alpha_2 \},$$

where  $\mathfrak{L}_X$  is the Lie derivative along the vector field  $X$  and

$$\alpha_2 = dx + \theta_1 d\theta_1 + \theta_2 d\theta_2.$$

Any contact vector field on  $\mathbb{R}^{1|2}$  can be expressed as

$$X_F = F\partial_x - \frac{1}{2}(-1)^{|F|} \sum_{i=1}^2 \bar{\eta}_i(F) \bar{\eta}_i, \quad \text{where } F \in C^\infty(\mathbb{R}^{1|2})$$

and  $\bar{\eta}_i = \partial_i - \theta_i \partial_x$ . The contact bracket is defined by  $[X_F, X_G] = X_{\{F, G\}}$ :

$$\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|} \sum_{i=1}^2 \bar{\eta}_i(F) \cdot \bar{\eta}_i(G). \quad (2.3)$$

The orthosymplectic Lie superalgebra  $\mathfrak{osp}(2|2)$  can be realized as a subalgebra of  $\mathcal{K}(2)$ :

$$\mathfrak{osp}(2|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta_1}, X_{x\theta_2}, X_{\theta_1}, X_{\theta_2}, X_{\theta_1\theta_2}).$$

We easily see that  $\mathfrak{osp}(1|2)$  is subalgebra of  $\mathfrak{osp}(2|2)$ :

$$\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta_1}, X_{\theta_1}) \simeq \text{Span}(X_1, X_x, X_{x^2}, X_{x\theta_2}, X_{\theta_2}).$$

We define the space of  $\lambda$ -densities as

$$\mathfrak{F}_\lambda^2 = \left\{ F(x, \theta_1, \theta_2) \alpha_2^\lambda \mid F(x, \theta_1, \theta_2) \in C^\infty(\mathbb{R}^{1|2}) \right\}. \quad (2.4)$$

As a vector space,  $\mathfrak{F}_\lambda^2$  is isomorphic to  $C^\infty(\mathbb{R}^{1|2})$ , but the Lie derivative of the density  $G\alpha_2^\lambda$  along the vector field  $X_F$  in  $\mathcal{K}(2)$  is now:

$$\mathfrak{L}_{X_F}(G\alpha_2^\lambda) = \mathfrak{L}_{X_F}^\lambda(G)\alpha_2^\lambda, \quad \text{with } \mathfrak{L}_{X_F}^\lambda(G) = \mathfrak{L}_{X_F}(G) + \lambda F'G. \quad (2.5)$$

Here, we restrict ourselves to the subalgebra  $\mathfrak{osp}(2|2)$ , thus we obtain a one-parameter family of  $\mathfrak{osp}(2|2)$ -modules on  $C^\infty(\mathbb{R}^{1|2})$  still denoted by  $\mathfrak{F}_\lambda^2$ . As an  $\mathfrak{osp}(1|2)$ -module, we have

$$\mathfrak{F}_\lambda^2 \simeq \mathfrak{F}_\lambda^1 \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1) \quad (2.6)$$

where  $\Pi$  is the change of parity operator.

Since  $-\bar{\eta}_i^2 = \partial_x$ , and  $\partial_i = \bar{\eta}_i - \theta_i \bar{\eta}_i^2$ , every differential operator  $A \in \mathfrak{D}_{\lambda, \mu}^2$  can be expressed in the form

$$A(F\alpha_2^\lambda) = \sum_{\ell, m} a_{\ell, m}(x, \theta) \bar{\eta}_1^\ell \bar{\eta}_2^m(F) \alpha_2^\mu, \quad (2.7)$$

where the coefficients  $a_{\ell, m}(x, \theta)$  are arbitrary functions.

**Proposition 2.1.** *As a  $\mathfrak{osp}(1|2)$ -module, we have*

$$\mathfrak{D}_{\lambda, \mu}^2 \simeq \mathfrak{D}_{\lambda, \mu}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}^1 \oplus \Pi \left( \mathfrak{D}_{\lambda, \mu+\frac{1}{2}}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2}, \mu}^1 \right). \quad (2.8)$$

Proof. Any element  $F \in C^\infty(\mathbb{R}^{1|2})$  can be uniquely written as follows:  $F = F_1 + F_2\theta_2$ , where  $\partial_2 F_1 = \partial_2 F_2 = 0$ . Therefore, for any  $X_H \in \mathfrak{osp}(1|2)$ , we easily check that

$$\mathfrak{L}_{X_H}^\lambda(F) = \mathfrak{L}_{X_H}^\lambda(F_1) + \mathfrak{L}_{X_H}^{\lambda+\frac{1}{2}}(F_2)\theta_2.$$

Thus, the following map is an  $\mathfrak{osp}(1|2)$ -isomorphism:

$$\begin{aligned} \Phi_\lambda : \quad \mathfrak{F}_\lambda^2 &\rightarrow \mathfrak{F}_\lambda^1 \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1) \\ F\alpha_2^\lambda &\mapsto \left( F_1\alpha_1^\lambda, \Pi(F_2\alpha_1^{\lambda+\frac{1}{2}}) \right) \end{aligned} \quad (2.9)$$

So, we deduce an  $\mathfrak{osp}(1|2)$ -isomorphism:

$$\begin{aligned} \Psi_{\lambda,\mu} : \quad \mathfrak{D}_{\lambda,\mu}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^1 \oplus \Pi\left(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^1 \oplus \mathfrak{D}_{\lambda+\frac{1}{2},\mu}^1\right) &\rightarrow \mathfrak{D}_{\lambda,\mu}^2 \\ A &\mapsto \Phi_\mu^{-1} \circ A \circ \Phi_\lambda. \end{aligned} \quad (2.10)$$

Here, we identify the  $\mathfrak{osp}(1|2)$ -modules via the following isomorphisms:

$$\begin{aligned} \Pi\left(\mathfrak{D}_{\lambda,\mu+\frac{1}{2}}^1\right) &\rightarrow \text{Hom}_{\text{diff}}\left(\mathfrak{F}_\lambda^1, \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^1)\right) & \Pi(A) &\mapsto \Pi \circ A, \\ \Pi\left(\mathfrak{D}_{\lambda+\frac{1}{2},\mu}^1\right) &\rightarrow \text{Hom}_{\text{diff}}\left(\mathfrak{F}_{\lambda+\frac{1}{2}}^1, \Pi(\mathfrak{F}_\mu^1)\right) & \Pi(A) &\mapsto A \circ \Pi, \\ \mathfrak{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^1 &\rightarrow \text{Hom}_{\text{diff}}\left(\Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1), \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^1)\right) & \Pi(A) &\mapsto \Pi \circ A \circ \Pi. \end{aligned}$$

□

### 3 The space $H^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda,\mu}^2)$

#### 3.1 Lie superalgebra cohomology, see [7]

Let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be a Lie superalgebra acting on a superspace  $V = V_0 \oplus V_1$  and let  $\mathfrak{k}$  be a subalgebra of  $\mathfrak{g}$ . (If  $\mathfrak{k}$  is omitted it assumed to be  $\{0\}$ .) The space of  $\mathfrak{k}$ -relative  $n$ -cochains of  $\mathfrak{g}$  with values in  $V$  is the  $\mathfrak{g}$ -module

$$C^n(\mathfrak{g}, \mathfrak{k}; V) := \text{Hom}_{\mathfrak{k}}(\Lambda^n(\mathfrak{g}/\mathfrak{k}); V).$$

The *coboundary operator*  $\delta_n : C^n(\mathfrak{g}, \mathfrak{k}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{k}; V)$  is a  $\mathfrak{g}$ -map satisfying  $\delta_n \circ \delta_{n-1} = 0$ . The kernel of  $\delta_n$ , denoted  $Z^n(\mathfrak{g}, \mathfrak{k}; V)$ , is the space of  $\mathfrak{k}$ -relative  $n$ -cocycles, among them, the elements in the range of  $\delta_{n-1}$  are called  $\mathfrak{k}$ -relative  $n$ -coboundaries. We denote  $B^n(\mathfrak{g}, \mathfrak{k}; V)$  the space of  $n$ -coboundaries.

By definition, the  $n^{\text{th}}$   $\mathfrak{k}$ -relative cohomology space is the quotient space

$$H^n(\mathfrak{g}, \mathfrak{k}; V) = Z^n(\mathfrak{g}, \mathfrak{k}; V) / B^n(\mathfrak{g}, \mathfrak{k}; V).$$

We will only need the formula of  $\delta_n$  (which will be simply denoted  $\delta$ ) in degrees 0 and 1: for  $v \in C^0(\mathfrak{g}, \mathfrak{k}; V) = V^\mathfrak{k}$ ,  $\delta v(g) := (-1)^{|g||v|} g \cdot v$ , where

$$V^\mathfrak{k} = \{v \in V \mid h \cdot v = 0 \text{ for all } h \in \mathfrak{k}\},$$

and for  $\Upsilon \in C^1(\mathfrak{g}, \mathfrak{k}; V)$ ,

$$\delta(\Upsilon)(g, h) := (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) - \Upsilon([g, h]) \quad \text{for any } g, h \in \mathfrak{g}.$$

### 3.2 The main theorem

The main result in this paper is the following:

**Theorem 3.1.** *The space  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)$  is purely even. It has the following structure:*

$$H_{\text{diff}}^1(\mathcal{K}(2), \mathfrak{D}_{\lambda, \mu}^2) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \mu - \lambda = 0, \\ \mathbb{R}^3 & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}) \text{ and } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.11)$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\begin{aligned} \Upsilon_{\lambda, \lambda}(X_G) &= G' \\ \tilde{\Upsilon}_{\lambda, \lambda}(X_G) &= \begin{cases} \bar{\eta}_1 \bar{\eta}_2(G) & \text{if } \lambda = 0 \\ 2\lambda \bar{\eta}_1(\partial_2(G)) - (-1)^{|G|}(\partial_2(G)\bar{\eta}_1 + \theta_2 \bar{\eta}_2 \bar{\eta}_1(G)\bar{\eta}_2) & \text{if } \lambda \neq 0 \end{cases} \\ \Upsilon_{-\frac{k}{2}, \frac{k}{2}}(X_G) &= G' \bar{\eta}_1 \bar{\eta}_2^{2k-1} \\ \tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}(X_G) &= k \bar{\eta}_1(\partial_2(G)) \bar{\eta}_1 \bar{\eta}_2^{2k-1} - (-1)^{|G|}(\partial_2(G) \bar{\eta}_2^{2k+1} - \bar{\eta}_1(\theta_2 \partial_2(G)) \bar{\eta}_1^{2k+1}) \\ \bar{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}(X_G) &= (k-1)G'' \bar{\eta}_1 \bar{\eta}_2^{2k-3} + (-1)^{|G|}(\bar{\eta}_2(G') \bar{\eta}_1^{2k-1} - \bar{\eta}_1(G') \bar{\eta}_2^{2k-1}) \end{aligned} \quad (3.12)$$

The proof of Theorem 3.1 will be the subject of Section 5. In fact, we need first the description of  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^2)$  and the  $\mathfrak{osp}(1|2)$ -relative cohomology  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}^2)$ . To describe the latter one, we need also the description of some bilinear  $\mathfrak{osp}(1|2)$ -invariant mappings.

## 4 Invariant Operators and Cohomology of $\mathfrak{osp}(1|2)$

### 4.1 Invariant bilinear differential operators

Observe that, as a  $\mathfrak{osp}(1|2)$ -module, we have

$$\mathfrak{osp}(2|2) \simeq \mathfrak{osp}(1|2) \oplus \Pi(\mathfrak{h}),$$

where  $\mathfrak{h}$  is the subspace of  $\mathfrak{F}_{-\frac{1}{2}}^1$  spanned by  $\{\theta_1 \alpha_1^{-\frac{1}{2}}, x \alpha_1^{-\frac{1}{2}}, \alpha_1^{-\frac{1}{2}}\}$ . In fact, it is easy to see that, for the adjoint action, the Lie superalgebra  $\mathcal{K}(2)$  is isomorphic to  $\mathfrak{F}_{-1}^2$  which is isomorphic, as  $\mathfrak{osp}(1|2)$ -module, to  $\mathfrak{F}_{-1}^1 \oplus \Pi(\mathfrak{F}_{-\frac{1}{2}}^1)$ . So, the space  $\mathfrak{osp}(2|2)$  is isomorphic, as a  $\mathfrak{osp}(1|2)$ -module, to  $\Phi_\lambda(\mathfrak{osp}(2|2))$ , where  $\Phi_\lambda$  is given by (2.9). More precisely, any element  $X_F$  is decomposed into  $X_F = X_{F_1} + X_{F_2 \theta_2}$  where  $\partial_2 F_1 = \partial_2 F_2 = 0$ , and then  $X_{F_1} \in \mathfrak{osp}(1|2)$  and  $X_{F_2 \theta_2}$  is identified to  $\Pi(F_2 \alpha_1^{-\frac{1}{2}}) \in \Pi(\mathfrak{h})$  and it will be denoted  $X_{\bar{F}_2}$ .

To compute the  $\mathfrak{osp}(1|2)$ -relative cohomology of  $\mathfrak{osp}(2|2)$ , we need the description of  $\mathfrak{osp}(1|2)$ -invariant mappings from  $\mathfrak{h} \otimes \mathfrak{F}_\lambda^1$  to  $\mathfrak{F}_\mu^1$ . To do that, we first, describe the  $\mathfrak{sl}(2)$ -invariant mappings from  $\mathfrak{h} \otimes \mathcal{F}_\lambda$  to  $\mathcal{F}_\mu$ . Obviously, as a  $\mathfrak{sl}(2)$ -module, we have  $\mathfrak{h} \simeq \mathfrak{h}_0 \oplus \Pi(\mathfrak{h}_1)$ , where  $\mathfrak{h}_0$  is the subspace of  $\mathcal{F}_{-\frac{1}{2}}$  spanned by  $\{x(dx)^{-\frac{1}{2}}, (dx)^{-\frac{1}{2}}\}$  and  $\mathfrak{h}_1$  is the subspace of  $\mathcal{F}_0$  spanned by 1.

**Lemma 4.1.** (see [3]) Let  $A : \mathfrak{h}_0 \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$ ,  $(hdx^{-\frac{1}{2}}, fdx^\lambda) \mapsto A(h, f)dx^\mu$  be an  $\mathfrak{sl}(2)$ -invariant nontrivial bilinear differential operator. Then  $\mu = \lambda + k - \frac{1}{2}$  where  $k$  is a non-negative integer satisfying

$$k(k-1)(2\lambda+k-1)(2\lambda+k-2) = 0,$$

and, up to a scalar factor, the map  $A$  is given by:

$$A(h, f) = hf^{(k)} + k(2\lambda+k-1)h'f^{(k-1)}.$$

By a straightforward computation, we can also check the following lemma.

**Lemma 4.2.** Let  $B : \mathfrak{h}_1 \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$ ,  $(h, fdx^\lambda) \mapsto B(h, f)dx^\mu$  be a nontrivial  $\mathfrak{sl}(2)$ -invariant bilinear differential operator, then

$$\mu = \lambda \quad \text{or} \quad (\lambda, \mu) = \left(\frac{1-k}{2}, \frac{1+k}{2}\right) \quad \text{and} \quad B(h, f) = ahf^{(\mu-\lambda)},$$

where  $k \in \mathbb{N}$  and  $a \in \mathbb{R}$ .

**Proposition 4.3.** Let  $\mathcal{A} : \mathfrak{h} \times \mathfrak{F}_\lambda^1 \rightarrow \mathfrak{F}_\mu^1$ ,  $(H\alpha_1^{-\frac{1}{2}}, F\alpha_1^\lambda) \mapsto \mathcal{A}(H, F)\alpha_1^\mu$  be a non-zero  $\mathfrak{osp}(1|2)$ -invariant bilinear differential operator. Then one of the following holds:

i) If  $\mu = \lambda + k - \frac{1}{2}$  where  $k$  is a non-negative integer satisfying  $k(k-1)(2\lambda+k-1) = 0$ , then, up to a scalar factor, the map  $\mathcal{A}$  is given by:

$$\mathcal{A}(H, F) = HF^{(k)} + k(2\lambda+k-1)H'F^{(k-1)} - (-1)^{|H|}k\overline{\eta}_1(H)\overline{\eta}_1(F^{(k-1)}). \quad (4.13)$$

ii) If  $\mu = \lambda + k$ , where  $k$  is a non-negative integer satisfying  $k(2\lambda+k)(2\lambda+k-1) = 0$ , then, up to a scalar factor, the map  $\mathcal{A}$  is given by:

$$\mathcal{A}(H, F) = (-1)^{|H|}H\overline{\eta}_1(F^{(k)}) + (2\lambda+k)\left(\overline{\eta}_1(H)F^{(k)} + kH'\overline{\eta}_1(F^{(k-1)})\right). \quad (4.14)$$

**Remark 4.4.** For  $k = 0, 1$ , the operators (4.13) and (4.14) are not only  $\mathfrak{osp}(1|2)$ -invariant, but also  $\mathcal{K}(1)$ -invariant.

Proof. Let  $A = A_{\bar{0}} + A_{\bar{1}}$  be the decomposition of  $A$  into even and odd parts. As  $\mathfrak{sl}(2)$ -module, we have

$$\mathfrak{h} \times \mathfrak{F}_\lambda^1 \simeq \mathfrak{h}_0 \otimes \mathcal{F}_\lambda \oplus \mathfrak{h}_0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) \oplus \Pi(\mathfrak{h}_1) \otimes \mathcal{F}_\lambda \oplus \Pi(\mathfrak{h}_1) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}). \quad (4.15)$$

So, the map  $A_{\bar{0}}$  is decomposed into four maps:

$$\begin{aligned} \mathfrak{h}_0 \otimes \mathcal{F}_\lambda &\longrightarrow \mathcal{F}_\mu, & \mathfrak{h}_0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\longrightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), \\ \Pi(\mathfrak{h}_1) \otimes \mathcal{F}_\lambda &\longrightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), & \Pi(\mathfrak{h}_1) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\longrightarrow \mathcal{F}_\mu \end{aligned} \quad (4.16)$$

and  $A_{\bar{1}}$  is also decomposed into four maps:

$$\begin{aligned} \mathfrak{h}_0 \otimes \mathcal{F}_\lambda &\longrightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}), & \mathfrak{h}_0 \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\longrightarrow \mathcal{F}_\mu, \\ \Pi(\mathfrak{h}_1) \otimes \mathcal{F}_\lambda &\longrightarrow \mathcal{F}_\mu, & \Pi(\mathfrak{h}_1) \otimes \Pi(\mathcal{F}_{\lambda+\frac{1}{2}}) &\longrightarrow \Pi(\mathcal{F}_{\mu+\frac{1}{2}}). \end{aligned} \quad (4.17)$$

Observe that the change of parity  $\Pi$  commutes with the  $\mathfrak{sl}(2)$ -action, therefore, according to Lemma 4.1 and Lemma 4.2, we can deduce the expressions of the operators (4.16) and (4.17). We conclude by using the invariance property with respect to  $X_{\theta_1}$  and  $X_{x\theta_1}$ .  $\square$

## 4.2 The space $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^2)$

Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Lie superalgebra, where  $\mathfrak{k}$  is a subalgebra and  $\mathfrak{p}$  is a  $\mathfrak{k}$ -module such that  $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ . Consider a 1-cocycle  $\Upsilon \in Z^1(\mathfrak{g}, V)$ , where  $V$  is a  $\mathfrak{g}$ -module. The cocycle relation reads

$$\Upsilon([g, h]) - (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) + (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) = 0, \quad g, h \in \mathfrak{g}.$$

Denote  $\Upsilon_{\mathfrak{k}} = \Upsilon|_{\mathfrak{k}}$  and  $\Upsilon_{\mathfrak{p}} = \Upsilon|_{\mathfrak{p}}$ . Obviously,  $\Upsilon_{\mathfrak{k}}$  is a 1-cocycle over  $\mathfrak{k}$  and if  $\Upsilon_{\mathfrak{k}} = 0$  then  $\Upsilon_{\mathfrak{p}}$  is  $\mathfrak{k}$ -invariant. Thus, the space  $H^1(\mathfrak{g}, V)$  is closely related to the space  $H^1(\mathfrak{k}, V)$ . Furthermore,  $\Upsilon_{\mathfrak{k}}$  and  $\Upsilon_{\mathfrak{p}}$  subject to the following equations:

$$\Upsilon_{\mathfrak{p}}([h, p]) - (-1)^{|h||\Upsilon|} h \cdot \Upsilon_{\mathfrak{p}}(p) + (-1)^{|p|(|h|+|\Upsilon|)} p \cdot \Upsilon_{\mathfrak{k}}(h) = 0, \quad h \in \mathfrak{k}, p \in \mathfrak{p}, \quad (4.18)$$

$$\Upsilon_{\mathfrak{k}}([p, p']) - (-1)^{|p||\Upsilon|} p \cdot \Upsilon_{\mathfrak{p}}(p') + (-1)^{|p'|(|p|+|\Upsilon|)} p' \cdot \Upsilon_{\mathfrak{p}}(p) = 0, \quad p, p' \in \mathfrak{p}. \quad (4.19)$$

In our situation,  $\mathfrak{g} = \mathfrak{osp}(2|2)$ ,  $\mathfrak{k} = \mathfrak{osp}(1|2)$ ,  $\mathfrak{p} = \Pi(\mathfrak{h})$  and  $V = \mathfrak{D}_{\lambda, \mu}^2$ . Thus, as a first step towards the proof of Theorem 3.1, we shall need to compute  $H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^2)$ . According to isomorphism (2.8), we can see that the knowledge of  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1)$  allows us to compute  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^2)$ :

$$\begin{aligned} H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^2) \simeq & H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1) \oplus H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda+\frac{1}{2}, \mu+\frac{1}{2}}^1) \oplus \\ & H_{\text{diff}}^1(\mathfrak{osp}(1|2), \Pi(\mathfrak{D}_{\lambda, \mu+\frac{1}{2}}^1)) \oplus H_{\text{diff}}^1(\mathfrak{osp}(1|2), \Pi(\mathfrak{D}_{\lambda+\frac{1}{2}, \mu}^1)). \end{aligned} \quad (4.20)$$

Of course, we can deduce the structure of  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \Pi(\mathfrak{D}_{\lambda, \mu}^1))$  from  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1)$ . Indeed, to any  $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1)$  corresponds  $\hat{\Upsilon} \in Z_{\text{diff}}^1(\mathfrak{osp}(1|2), \Pi(\mathfrak{D}_{\lambda, \mu}^1))$  where  $\hat{\Upsilon}(X_G) = \Pi(\sigma \circ \Upsilon(X_G))$  with  $\sigma(F) = (-1)^{|F|} F$ . Obviously,  $\Upsilon$  is a coboundary if and only if  $\hat{\Upsilon}$  is a coboundary. Thus, we recall the space  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1)$  which was computed in [3]:

$$H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1) \simeq \begin{cases} \mathbb{R} & \text{if } \lambda = \mu, \\ \mathbb{R}^2 & \text{if } \lambda = \frac{1-k}{2}, \mu = \frac{k}{2}, k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

A basis for the space  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}^1)$  is given by the cohomology classes of the 1-cocycles  $\Gamma_{\lambda, \mu}$  and  $\tilde{\Gamma}_{\lambda, \mu}$  defined by:

$$\begin{aligned} \Gamma_{\lambda, \lambda}(X_G) &= G', \\ \Gamma_{\frac{1-k}{2}, \frac{k}{2}}(X_G) &= (-1)^{|G|} \bar{\eta}_1^2(G) \bar{\eta}_1^{2k-1}, \\ \tilde{\Gamma}_{\frac{1-k}{2}, \frac{k}{2}}(X_G) &= (-1)^{|G|} (k-1) \bar{\eta}_1^4(G) \bar{\eta}_1^{2k-3} + \bar{\eta}_1^3(G) \bar{\eta}_1^{2k-2}. \end{aligned} \quad (4.21)$$

## 4.3 The space $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}^2)$

In this subsection we compute the space  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}^2)$  and we prove that it is nontrivial which is not the case for  $n = 1$ :  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{sl}(2); \mathfrak{D}_{\lambda, \mu}^1) = 0$ , see [3]. Moreover, the first author, in [6], proved that the space  $H_{\text{diff}}^1(\mathcal{K}(2), \mathcal{K}(1); \mathfrak{D}_{\lambda, \mu}^2)$  is nontrivial while the space  $H_{\text{diff}}^1(\mathcal{K}(1), \text{Vect}(\mathbb{R}); \mathfrak{D}_{\lambda, \mu}^1)$  is trivial, see [4]. Hence, the case  $n = 2$  appears as a special case.

**Theorem 4.5.**  $\dim H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda, \mu}^2) \leq 1$ . It is 1 only if  $\lambda = \mu \neq 0$  or  $(\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2})$ , where  $k \in \mathbb{N} \setminus \{0\}$ . The cohomology classes of the following 1-cocycles generate the corresponding spaces:

$$\begin{aligned}\tilde{\Upsilon}_{\lambda, \lambda}(X_G) &= 2\lambda \bar{\eta}_1(\partial_2(G)) - (-1)^{|G|}(\partial_2(G)\bar{\eta}_1 + \theta_2\bar{\eta}_2\bar{\eta}_1(G)\bar{\eta}_2), \\ \tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}(X_G) &= k\bar{\eta}_1(\partial_2(G))\bar{\eta}_1\bar{\eta}_2^{2k-1} - (-1)^{|G|}(\partial_2(G)\bar{\eta}_2^{2k+1} - \bar{\eta}_1(\theta_2\partial_2(G))\bar{\eta}_1^{2k+1}).\end{aligned}\tag{4.22}$$

To prove Theorem 4.5, we need the following classical fact:

**Lemma 4.6.** Let  $\mathfrak{g}$  be a Lie superalgebra and  $A : U \otimes V \rightarrow W$  a bilinear map, where  $U, V$  and  $W$  are  $\mathfrak{g}$ -modules. We consider the following associated maps

$$\begin{aligned}A_1 : \Pi(U) \otimes V &\rightarrow W, & A_2 : \Pi(U) \otimes \Pi(V) &\rightarrow \Pi(W), \\ A_3 : \Pi(U) \otimes V &\rightarrow \Pi(W), & A_4 : \Pi(U) \otimes \Pi(V) &\rightarrow W\end{aligned}$$

defined by

$$\begin{aligned}A_1(\Pi(u) \otimes v) &= (-1)^{|u|}A(u \otimes v), & A_2(\Pi(u) \otimes \Pi(v)) &= (-1)^{|u|}\Pi(A(u \otimes v)), \\ A_3(\Pi(u) \otimes v) &= (-1)^{|v|}\Pi(A(u \otimes v)), & A_4(\Pi(u) \otimes \Pi(v)) &= (-1)^{|v|}A(u \otimes v).\end{aligned}$$

The maps  $A_1, A_2, A_3$  and  $A_4$  are  $\mathfrak{g}$ -invariant if and only if  $A$  is  $\mathfrak{g}$ -invariant.

Proof.(Theorem 4.5): Consider a 1-cocycle  $\Upsilon$  over  $\mathfrak{osp}(2|2)$  vanishing on  $\mathfrak{osp}(1|2)$ . Thus, the equations (4.18) and (4.19) become

$$X_G \cdot \Upsilon(X_{\bar{H}}) - (-1)^{|G||\Upsilon|}\Upsilon([X_G, X_{\bar{H}}]) = 0,\tag{4.23}$$

$$(-1)^{|\bar{H}_1||\Upsilon|}X_{\bar{H}_1} \cdot \Upsilon(X_{\bar{H}_2}) - (-1)^{(|\bar{H}_1|+|\Upsilon|)|\bar{H}_2|}X_{\bar{H}_2} \cdot \Upsilon(X_{\bar{H}_1}) = 0,\tag{4.24}$$

for all  $X_{\bar{H}}, X_{\bar{H}_1}, X_{\bar{H}_2} \in \Pi(\mathfrak{h})$  and  $X_G \in \mathfrak{osp}(1|2)$ . According to the isomorphism (2.9), the map  $\Upsilon$  is decomposed into four components:

$$\begin{aligned}\Pi(\mathfrak{h}) \times \mathfrak{F}_{\lambda}^1 &\rightarrow \mathfrak{F}_{\mu}^1, & \Pi(\mathfrak{h}) \times \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1) &\rightarrow \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^1), \\ \Pi(\mathfrak{h}) \times \mathfrak{F}_{\lambda}^1 &\rightarrow \Pi(\mathfrak{F}_{\mu+\frac{1}{2}}^1), & \Pi(\mathfrak{h}) \times \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1) &\rightarrow \mathfrak{F}_{\mu}^1.\end{aligned}\tag{4.25}$$

The equation (4.23) expresses the  $\mathfrak{osp}(1|2)$ -invariance of each of these bilinear maps. Thus, using Proposition 4.3, Lemma 4.6 and equation (4.24), we prove that, if  $\Upsilon$  is an odd 1-cocycle then, up to a scalar factor,  $\Upsilon$  is given by ( with  $a, b \in \mathbb{R}$  and  $k \in \mathbb{N} \setminus \{0\}$ ):

$$\Upsilon = \begin{cases} \delta(a\partial_2^k + b(\bar{\eta}_1 + \theta_2\bar{\eta}_1\bar{\eta}_2)\partial_x^{k-1}) & \text{if } (\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2}), \\ \delta(a\partial_2^k + b\theta_2\bar{\eta}_1\bar{\eta}_2\partial_x^{k-1}) & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k-1}{2}), \\ \delta(\partial_2) & \text{if } \mu = \lambda + \frac{1}{2} \text{ and } \lambda \neq 0, -\frac{1}{2}, \\ \delta(\theta_2) & \text{if } \mu = \lambda - \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$



Now, if  $\Upsilon$  is an even 1-cocycle, by the same arguments as above, we get:

$$\Upsilon = \begin{cases} a\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}} + b\delta(\bar{\eta}_1\partial_2\partial_x^{k-1}) & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}), \\ a\tilde{\Upsilon}_{\lambda, \lambda} + b\delta(\theta_2\bar{\eta}_2) & \text{if } \lambda = \mu \neq 0, \\ \delta(\theta_2(a\bar{\eta}_1 + b\bar{\eta}_2)) & \text{if } \lambda = \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

where  $\tilde{\Upsilon}_{\lambda, \lambda}$  and  $\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}$  are those given in (4.22). Therefore, in order to complete the proof of Theorem 4.5, we have to study the cohomology classes of the 1-cocycles  $\tilde{\Upsilon}_{\lambda, \lambda}$  and  $\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}$ .

**Lemma 4.7.** *The maps  $\tilde{\Upsilon}_{\lambda, \lambda}$  and  $\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}$  are nontrivial  $\mathfrak{osp}(1|2)$ -relative 1-cocycles.*

Proof. First, we can easily see that, for any even element  $F \in C^\infty(\mathbb{R}^{1|2})$ ,

$$\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}(X_{\theta_1\theta_2})(F\alpha_2^{-\frac{k}{2}}) = -k\bar{\eta}_1\bar{\eta}_2^{2k-1}(F)\alpha_2^{\frac{k}{2}} \quad (4.26)$$

Next, assume that there exists an even operator  $A \in \mathfrak{D}_{-\frac{k}{2}, \frac{k}{2}}^2$  such that  $\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}$  is equal to  $\delta A$ , that is

$$\tilde{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}(X_G) = \mathfrak{L}_{X_G}^{\frac{k}{2}} \circ A - A \circ \mathfrak{L}_{X_G}^{-\frac{k}{2}}. \quad (4.27)$$

The operator  $A$  is of the form (2.7); the condition (4.27) implies that its coefficients are constants (which is equivalent to the fact that  $X_1 \cdot A = 0$ ). Then, it is now easy to check that the condition (4.27) has no solution: using formula (2.5), we can see that the expression (4.26) never appear in the right hand side of (4.27). This is a contradiction with our assumption. Similarly, we prove that the cocycle  $\tilde{\Upsilon}_{\lambda, \lambda}$  is nontrivial. Lemma 4.7 is proved. Thus we have completed the proof of Theorem 4.5.  $\square$

**Corollary 4.8.** *Up to a coboundary, any 1-cocycle  $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)$  is invariant with respect to the vector field  $X_1 = \partial_x$ . That is, the map  $\Upsilon$  can be expressed with constant coefficients.*

Proof. The 1-cocycle condition reads:

$$X_1 \cdot \Upsilon(X_F) - (-1)^{|F||\Upsilon|} X_F \cdot \Upsilon(X_1) - \Upsilon([X_1, X_F]) = 0. \quad (4.28)$$

But, from (4.21) and Theorem 4.5, it follows that, up to a coboundary, we have  $\Upsilon(X_1) = 0$ , and therefore the equation (4.28) becomes

$$X_1 \cdot \Upsilon(X_F) - \Upsilon([X_1, X_F]) = 0$$

which is nothing but the invariance property of  $\Upsilon$  with respect to  $X_1$ .  $\square$

## 5 Proof of Theorem 3.1

Consider a 1-cocycle  $\Upsilon \in Z_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)$ . If  $\Upsilon|_{\mathfrak{osp}(1|2)}$  is trivial then the 1-cocycle  $\Upsilon$  is completely described by Theorem 4.5. Thus, assume that  $\Upsilon|_{\mathfrak{osp}(1|2)}$  is nontrivial. Of course, up to coboundary, the general form of  $\Upsilon|_{\mathfrak{osp}(1|2)}$  is given by (4.21) together with the isomorphism

(4.20) while  $\Upsilon_{|\Pi(\mathfrak{h})}$  can be essentially described by equation (4.18) and Corollary 4.8. More precisely, according to (4.21) and the isomorphism (4.20), the 1-cocycle  $\Upsilon$  can be nontrivial only, a priori, if  $\lambda = \mu$ , or  $\lambda = \mu \pm \frac{1}{2}$ , or  $(\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}), (\frac{1-k}{2}, \frac{k}{2}), (-\frac{k}{2}, \frac{k-1}{2})$  where  $k \in \mathbb{N}$ . Thus, we have to distinguish all these cases. Hereafter all  $\varepsilon$ 's are constants.

**The case where  $\lambda = \mu$**

Considering (4.21) and the isomorphism (4.20), we see that there are two subcases:

i)  $\lambda = \mu \neq 0$ . In this case, the map  $\Upsilon_{|\mathfrak{osp}(1|2)}$  is, a priori, given by

$$\Upsilon_{|\mathfrak{osp}(1|2)}(X_{G_1})(F\alpha_2^\lambda) = (\varepsilon_1 G'_1 F_1 + \varepsilon_2 G'_1 F_2 \theta_2) \alpha_2^\lambda,$$

where  $F = F_1 + F_2 \theta_2$ , with  $\partial_2 F_1 = \partial_2 F_2 = 0$ . By direct computation, using equations (4.18)–(4.19) and Corollary 4.8, we deduce that

$$\varepsilon_1 = \varepsilon_2 \quad \text{and} \quad \Upsilon_{|\Pi(\mathfrak{h})}(X_{\bar{G}_2})(F\alpha_2^\lambda) = \varepsilon_1 (-1)^{|F|} G'_2 F \theta_2 \alpha_2^\lambda.$$

Hence  $\Upsilon$  is a multiple of  $\Upsilon_{\lambda, \lambda}$ , see (3.12).

ii)  $\lambda = \mu = 0$ . Here the map  $\Upsilon_{|\mathfrak{osp}(1|2)}$  is, a priori, given by

$$\Upsilon_{|\mathfrak{osp}(1|2)}(X_{G_1})(F) = \left( \varepsilon_1 G'_1 F_1 + \left( \varepsilon_2 G'_1 F_2 + (-1)^{|F_1|} (\varepsilon_3 G'_1 \bar{\eta}_1(F_1) + \varepsilon_4 \eta_1(G'_1) F_1) \right) \theta_2 \right).$$

The same arguments, as above, show that  $\varepsilon_1 = \varepsilon_2$ ,  $\varepsilon_3 = 0$  and

$$\Upsilon_{|\Pi(\mathfrak{h})}(X_{\bar{G}_2})(F) = \left( \varepsilon_1 G'_2 \theta_2 + \varepsilon_4 (-1)^{|G_2|} \bar{\eta}_1(G_2) \right) F.$$

Hence  $\Upsilon$  is linear combination of  $\Upsilon_{0,0}$  and  $\tilde{\Upsilon}_{0,0}$ , see (3.12).

**The case where  $\mu - \lambda = k$  and  $2\lambda = -k \neq 0$**

In this case, the map  $\Upsilon_{|\mathfrak{osp}(1|2)}$  is, a priori, given by

$$\begin{aligned} \Upsilon_{|\mathfrak{osp}(1|2)}(X_{G_1})(F\alpha_2^{-\frac{k}{2}}) &= \left( (-1)^{|F_1|} \left( \varepsilon_1 G'_1 \bar{\eta}_1(F_1^{(k)}) + \varepsilon_2 \left( k G'_1 \bar{\eta}_1^{2k-1}(F_1) + \eta_1(G'_1) \bar{\eta}_1^{2k}(F_1) \right) \right) \theta_2 \right. \\ &\quad \left. + (-1)^{|F_2|} \left( \varepsilon_3 \left( (k-1) G'_1 \bar{\eta}_1^{2k-3}(F_2) + \eta_1(G'_1) \bar{\eta}_1^{2k-2}(F_2) \right) \right. \right. \\ &\quad \left. \left. + \varepsilon_4 G'_1 \bar{\eta}_1(F_2^{(k-1)}) \right) \right) \alpha_2^{\frac{k}{2}}. \end{aligned}$$

Again by the same arguments, we prove that we have  $\varepsilon_4 = -\varepsilon_1$ ,  $\varepsilon_3 = \varepsilon_2$  and

$$\Upsilon_{|\Pi(\mathfrak{h})}(X_{\bar{G}_2})(F\alpha_2^{-\frac{k}{2}}) = \left( \varepsilon_1 G'_2 \bar{\eta}_1(F_2^{(k-1)}) \theta_2 - \varepsilon_2 G'_2 \bar{\eta}_1^{2k-1}(F) \right) \alpha_2^{\frac{k}{2}}.$$

Hence  $\Upsilon$  is linear combination of  $\Upsilon_{-\frac{k}{2}, \frac{k}{2}}$  and  $\bar{\Upsilon}_{-\frac{k}{2}, \frac{k}{2}}$ , see (3.12).

For the cases where  $\lambda = \mu \pm \frac{1}{2}$ , or  $(\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2}), (-\frac{k}{2}, \frac{k-1}{2})$ , the same arguments as before, show that  $\Upsilon$  is trivial. This completes the proof.  $\square$

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